Homework 8
Geometry

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Proposition 0.1 (Exercise 9-19)**.** *Let M be $\mathbb{R}^3$ with the z-axis removed. Define smooth vector fields V and W on M by*

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z} \qquad W = \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}$$

and let θ be the flow of V and ψ be the flow of W. Then V, W commute, but there exist $p \in M$ and $s, t \in \mathbb{R}$ so that $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are both defined but are not equal.

Proof. First we show that V, W commute by showing that $[V, W] = 0$.

$$\begin{aligned}
[V, W] &= \left[\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\right] \\
&= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + \left[\frac{\partial}{\partial x}, \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\right] - \left[\frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right] - \left[\frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\right] \\
&= 0 + \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}\right)\frac{\partial}{\partial z} + \left(\frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)\frac{\partial}{\partial z} + 0 \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}\frac{\partial}{\partial z} + \frac{x^2-y^2}{(x^2+y^2)^2}\frac{\partial}{\partial z} \\
&= 0
\end{aligned}$$

Thus V and W commute. Now we let $p = (1, 0, 0)$ and $s = t = 1$ and compute $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$. We begin by computing $\theta_1(p)$. An integral curve $\gamma(t) = (x(t), y(t), z(t))$ of V through p satisfies the the system of differential equations

$$\begin{aligned}
\dot{x} &= 1 \\
\dot{y} &= 0 \\
\dot{z} &= \frac{-y}{x^2+y^2}
\end{aligned}$$

with initial condition $(x, y, z)(0) = (1, 0, 0)$. The solution is given by

$$\begin{aligned}
x(t) &= t + 1 \\
y(t) &= 0 \\
z(t) &= 0
\end{aligned}$$

thus $\theta_1(1,0,0) = (2,0,0)$. Now we compute $\psi_1(p)$. This involves solving the system

$$\begin{aligned}\dot{u} &= 0\\ \dot{v} &= 1\\ \dot{w} &= \frac{u}{u^2+v^2}\end{aligned}$$

We can immediately see that $u(t) = 1$ and $v(t) = t$, which means that we have

$$w(t) = \int \frac{1}{1+t^2}dt = \arctan(t) + C$$

Given the initial condition $w(0) = 0$, we have $C = 0$. Thus the solution to the IVP is

$$\begin{aligned}u(t) &= 1\\ v(t) &= t\\ w(t) &= \arctan(t)\end{aligned}$$

Thus $\psi_1(p) = (1, 1, \arctan(1)) = (1, 1, \frac{\pi}{4})$. Now we compute $\psi_1 \circ \theta_1(p)$. We solve the same system in u, v, w as above, now with the different initial condition $(u, v, w)(0) = (2, 0, 0)$. The solution is

$$\begin{aligned}u(t) &= 2\\ v(t) &= t+1\\ w(t) &= \arctan\left(\frac{t+1}{2}\right) - \arctan\left(\frac{1}{2}\right)\end{aligned}$$

thus $\psi_1 \circ \theta_1(p) = \left(2, 2, \arctan(1) - \arctan(\frac{1}{2})\right)$. Finally, we compute $\theta_1 \circ \psi_1(p)$. We solve the previous system in x, y, z with initial condition $(x, y, z)(0) = (1, 1, \frac{\pi}{4})$. We can see immediately that $y(t) = 1$ and $x(t) = t + C$. Using our initial condition, $x(0) = 1 = C$. Then we have

$$z(t) = \int \frac{-1}{(t+1)^2+1}dt = -\arctan(t+1) + C$$

and using our initial condition $z(0) = \frac{\pi}{4} = -\arctan(1) + C$ we get $C = \frac{\pi}{2}$ so

$$\begin{aligned}x(t) &= t+1\\ y(t) &= 1\\ z(t) &= -\arctan(t+1) + \frac{\pi}{2}\end{aligned}$$

Thus $\theta_1 \circ \psi_1(p) = \left(2, 1, -\arctan(2) + \frac{\pi}{2}\right)$. Recall that we computed

$$\psi_1 \circ \theta_1(p) = \left(2, 2, \arctan(1) - \arctan\left(\frac{1}{2}\right)\right)$$

The second entries are obviously not equal (nor are the third entries, though that is less obvious), so our claim is proven. $\square$

(Exercise 10-7)
Compute the transition function for TS^2 associated with the two local trivializations determined by stereographic coordinates.

Solution. We denote the stereographic coordinates by $\phi = (x, y)$ and $\psi = (u, v)$, where $\phi : S^2 \setminus \{N\} \to \mathbb{R}^2$ and $\psi : S^2 \setminus \{S\} \to \mathbb{R}^2$, given explicitly by

$$(x, y) = \phi(p) = \phi(p^1, p^2, p^3) = \left(\frac{p^1}{1 - p^3}, \frac{p^2}{1 - p^3}\right)$$
$$(u, v) = \psi(p) = \psi(p^1, p^2, p^3) = \left(\frac{p^1}{1 + p^3}, \frac{p^2}{1 + p^3}\right)$$

For the tangent bundle $\pi : TS^2 \to S$ given by $(p, v) \mapsto p$ these charts give local trivializations

$$\Phi : \pi^{-1}(S^2 \setminus \{N\}) \to (S^2 \setminus \{N\}) \times \mathbb{R}^2$$
$$\Psi : \pi^{-1}(S^2 \setminus \{S\}) \to (S^2 \setminus \{S\}) \times \mathbb{R}^2$$

Explicitly, these are given by

$$\Phi\left(w^1 \frac{\partial}{\partial x}\bigg|_p + w^2 \frac{\partial}{\partial y}\bigg|_p\right) = (p, (w^1, w^2))$$
$$\Psi\left(w^1 \frac{\partial}{\partial u}\bigg|_p + w^2 \frac{\partial}{\partial v}\bigg|_p\right) = (p, (w^1, w^2))$$

The transition function associated with these local trivializations is a map

$$\tau : S^2 \setminus \{N, S\} \to \mathrm{GL}(2, \mathbb{R})$$

such that

$$\Phi \circ \Psi^{-1}(p, w) = (p, \tau(p)w)$$

where w is the column vector (w^1, w^2). In Exercise 1-7, we computed the transition map between the charts ϕ and ψ to be

$$(x, y) = \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right)$$

and so we can compute all the entries of the Jacobian:

$$\frac{\partial x}{\partial u} = \frac{v^2 - u^2}{(u^2 + v^2)^2} \qquad \frac{\partial x}{\partial v} = \frac{-2uv}{(u^2 + v^2)^2} \qquad \frac{\partial y}{\partial u} = \frac{-2uv}{(u^2 + v^2)^2} \qquad \frac{\partial y}{\partial v} = \frac{u^2 - v^2}{(u^2 + v^2)^2}$$

This allows us to do the following change of coordinates explicitly.

$$\frac{\partial}{\partial u}\bigg|_p = \frac{\partial x}{\partial u}\frac{\partial}{\partial x}\bigg|_p + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}\bigg|_p$$
$$\frac{\partial}{\partial v}\bigg|_p = \frac{\partial x}{\partial v}\frac{\partial}{\partial x}\bigg|_p + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}\bigg|_p$$

Finally, we compute the transition function $\Phi \circ \Psi^{-1}(p, w)$ to compute $\tau(p)$.

$$
\begin{aligned}
\Phi \circ \Psi^{-1}(p, w) &= \Phi\left(w^1 \frac{\partial}{\partial u}\bigg|_p + w^2 \frac{\partial}{\partial v}\bigg|_p \right) \\
&= \Phi\left(w^1 \left(\frac{\partial x}{\partial u} \frac{\partial}{\partial x}\bigg|_p + \frac{\partial y}{\partial u} \frac{\partial}{\partial y}\bigg|_p \right) + v^2 \left(\frac{\partial x}{\partial v} \frac{\partial}{\partial x}\bigg|_p + \frac{\partial y}{\partial v} \frac{\partial}{\partial y}\bigg|_p \right) \right) \\
&= \Phi\left(\left(v^1 \frac{\partial x}{\partial u} + v^2 \frac{\partial x}{\partial v} \right) \frac{\partial}{\partial x}\bigg|_p + \left(v^1 \frac{\partial y}{\partial u} + v^2 \frac{\partial y}{\partial v} \right) \frac{\partial}{\partial y}\bigg|_p \right) \\
&= \left(p, \left(v^1 \frac{\partial x}{\partial u} + v^2 \frac{\partial x}{\partial v}, v^1 \frac{\partial y}{\partial u} + v^2 \frac{\partial y}{\partial v} \right) \right) \\
&= \left(p, \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \right)
\end{aligned}
$$

Thus

$$
\tau(p) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{v^2 - u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{-2uv}{(u^2+v^2)^2} & \frac{u^2 - v^2}{(u^2+v^2)^2} \end{bmatrix}
$$

where (u, v) is a function of p as above, so we simplify. We switch to subscripts for indices so that we can use superscripts for exponents. Because $(p_1, p_2, p_3) \in S^2$, we have $p_1^2 + p_2^2 + p_3^2 = 1$ so $p_1^2 + p_2^2 = 1 - p_3^2 = (1 - p_3)(1 + p_3)$. In terms of p, the entries of $\tau(p)$ are

$$
\begin{aligned}
\frac{v^2 - u^2}{(u^2 + v^2)^2} &= \frac{-(p_1^2 - p_2^2)(1 + p_3)^2}{(p_1^2 + p_2^2)^2} = \frac{-(p_1 - p_2)(p_1 + p_2)(1 + p_3)}{(1 - p_3)} \\
\frac{u^2 - v^2}{(u^2 + v^2)^2} &= \frac{(p_1^2 - p_2^2)(1 + p_3)^2}{(p_1^2 + p_2^2)^2} = \frac{(p_1 - p_2)(p_1 + p_2)(1 + p_3)}{(1 - p_3)} \\
\frac{-2uv}{(u^2 + v^2)^2} &= \frac{-2p_1 p_2 (1 + p_3)^2}{(p_1^2 + p_2^2)^2} = \frac{-2p_1 p_2 (1 + p_3)}{(1 - p_3)}
\end{aligned}
$$

(Recall that τ is a map only on $S^2 \setminus \{N, S\}$ so we never have $p_3 = 1$, so the denominators are never zero.)

Proposition 0.2 (Exercise 10-12). *Let M be a smooth manifold with or without boundary and let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be two smooth rank-k vector bundles over M. Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}, \{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$ respectively. Then $E, \widetilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that*

$$
\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p) \tau_{\alpha\beta}(p) \sigma_\beta(p)^{-1}
$$

for all $p \in U_\alpha \cap U_\beta$.

Proof. First ssppose that $E, \widetilde{E}$ are smoothly isomorphic over M. Let $\alpha \in A$, and let $\phi_\alpha, \widetilde{\phi}_\alpha$ be local trivializations of $E, \widetilde{E}$ respectively and $\tau_{\alpha\beta}, \widetilde{\tau}_{\alpha\beta}$ be the transition maps. Let $F : E \to \widetilde{E}$ be a bundle isomorphism. Then we have

$$\pi = \widetilde{\pi} \circ F \qquad \widetilde{\pi} = \pi_{U_\alpha} \circ \widetilde{\phi}_\alpha \qquad \pi = \pi_{U_\alpha} \circ \phi_\alpha$$

We define ψ_α by $\psi_\alpha = \widetilde{\phi}_\alpha \circ F$ and claim that ψ_α is a local trivialization of E over U_α. The condition $\pi_{U_\alpha} \circ \psi_\alpha = \pi$ is satisfied as we have

$$\pi_{U_\alpha} \circ \psi_\alpha = \pi_{U_\alpha} \circ \widetilde{\phi}_\alpha \circ F = \widetilde{\pi} \circ F = \pi$$

We need to show that the restriction of ψ_α to $E_q = \pi^{-1}(q)$ is a vector space isomorphism to $\{q\} \times \mathbb{R}^k$. Note that $\pi^{-1} = F^{-1} \circ \widetilde{\pi}^{-1}$, so

$$\psi_\alpha(E_q) = \widetilde{\phi}_\alpha \circ F(E_q) = \widetilde{\phi}_\alpha \circ F \circ \pi^{-1}(q) = \widetilde{\phi}_\alpha \circ F \circ F^{-1} \circ \widetilde{\pi}^{-1}(q) = \widetilde{\phi}_\alpha \circ \widetilde{\pi}^{-1}(q)$$

Since $\widetilde{\phi}_\alpha$ is a local trivialization of $\widetilde{E}$, we have what we needed: $\psi_\alpha(E_q) = \widetilde{\phi}_\alpha \circ \widetilde{\pi}^{-1}(q)$ is isomorphic to $\{q\} \times \mathbb{R}^k$.

Now we have smooth local trivializations ϕ_α and ψ_α of E over U_α for each α. By Lemma 10.5, there exist smooth maps $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ and $\sigma_\beta : U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\begin{aligned} \psi_\alpha \circ \phi_\alpha^{-1}(p, v) &= (p, \sigma_\alpha(p)v) \\ \psi_\beta \circ \phi_\beta^{-1}(p, v) &= (p, \sigma_\beta(p)v) \end{aligned}$$

Now we do a long computation to show that $\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}$.

$$\begin{aligned} (p, \widetilde{\tau}_{\alpha\beta}(p)v) &= \widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p, v) \\ &= \widetilde{\phi}_\alpha \circ (F \circ F^{-1}) \circ \widetilde{\phi}_\beta^{-1}(p, v) \\ &= \widetilde{\phi}_\alpha \circ F \circ (\phi_\alpha^{-1} \circ \phi_\alpha) \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ F^{-1} \circ \widetilde{\phi}_\beta^{-1}(p, v) \\ &= \psi_\alpha \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1} \circ \psi_\beta^{-1}(p, v) \\ &= (\psi_\alpha \circ \phi_\alpha)^{-1} \circ (\phi_\alpha \circ \phi_\beta^{-1}) \circ (\psi_\beta \circ \phi_\beta^{-1})^{-1}(p, v) \\ &= (\psi_\alpha \circ \phi_\alpha)^{-1}(\phi_\alpha \circ \phi_\beta^{-1})(p, \sigma_\beta(p)^{-1}v) \\ &= (\psi_\alpha \circ \phi_\alpha)(p, \tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}v) \\ &= (p, \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}v) \end{aligned}$$

Thus $\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}$. Now we show the other direction of implication. Suppose that for each α, the required σ_α exists. We must construct a bundle isomorphism $F : E \to \widetilde{E}$. First, we define $f_\alpha : U_\alpha \times \mathbb{R}^k \to U_\alpha \times \mathbb{R}^k$ by $f_\alpha(p, v) = \sigma_\alpha(p)v)$. Then we define $F_\alpha : \pi^{-1}(U_\alpha) \to \widetilde{\pi}^{-1}(U_\alpha)$ by $F_\alpha = \widetilde{\phi}_\alpha^{-1} \circ f_\alpha \circ \phi_\alpha$. Then we check that F_α agrees with F_β on the overlap $U_\alpha \cap U_\beta$:

$$\begin{aligned} F_\beta &= \widetilde{\phi}_\beta^{-1} \circ f_\beta \circ \phi_\beta \\ &= (\widetilde{\phi}_\alpha^{-1} \circ \widetilde{\phi}_\alpha) \circ \widetilde{\phi}_\beta^{-1} \circ f_\beta \circ \phi_\beta \circ (\phi_\alpha^{-1} \circ \phi_\alpha) \\ &= \widetilde{\phi}_\alpha^{-1} \circ (\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}) \circ f_\beta \circ (\phi_\beta \circ \phi_\alpha^{-1}) \circ \phi_\alpha \end{aligned}$$

Now note that

$$\begin{aligned}(\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}) \circ f_\beta \circ (\phi_\beta \circ \phi_\alpha^{-1})(p, v) &= (\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}) \circ f_\beta(p, \widetilde{\tau}_{\alpha\beta}(p)^{-1}v) \\ &= (\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1})(p, \sigma_\beta \tau_{\alpha\beta}(p)^{-1}v) \\ &= (p, \widetilde{\tau}_{\alpha\beta}(p)\sigma_\beta(p)\tau_{\alpha\beta}(p)^{-1}v)\end{aligned}$$

By hypothesis, $\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}$, so $\widetilde{\tau}_{\alpha\beta}(p)\sigma_\beta(p)\tau_{\alpha\beta}(p)^{-1} = \sigma_\alpha(p)$. Thus

$$(\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}) \circ f_\beta \circ (\phi_\beta \circ \phi_\alpha^{-1})(p, v) = (p, \sigma_\alpha(p)v) = f_\alpha(p, v)$$

Thus

$$F_\beta = \widetilde{\phi}_\alpha^{-1} \circ (\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}) \circ f_\beta \circ (\phi_\beta \circ \phi_\alpha^{-1}) \circ \phi_\alpha = \widetilde{\phi}_\alpha^{-1} f_\alpha \phi_\alpha = F_\alpha$$

on the overlaps $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$. Clearly F is smooth as a composition other smooth functions. Thus by Corollary 2.8, there is a unique smooth map $F : E \to \widetilde{E}$ that agrees with F_α on $\pi^{-1}(U_\alpha)$. We claim this map is a bundle isomorphism. First, we show that $\widetilde{\pi} \circ F = \pi$.

$$\widetilde{\pi} \circ F = \widetilde{\pi} \circ \widetilde{\phi}_\alpha^{-1} \circ f_\alpha \circ \phi_\alpha = \pi_{U_\alpha} \circ f_\alpha \circ \phi_\alpha = \pi_{U_\alpha} \circ \phi_\alpha = \pi$$

Note that $\pi_{U_\alpha} \circ f_\alpha(p, v) = \pi_{U_\alpha}(p, \sigma_\alpha(p)v) = p = \pi_{U_\alpha}(p, v)$. To see that $F|_{E_q}$ is linear, note that

$$F|_{E_q} = \widetilde{\phi}_\alpha^{-1} \circ f_\alpha \circ \phi_\alpha|_{E_q}$$

Because they are trivializations, $\widetilde{\phi}_\alpha^{-1}$ and ϕ_α are vector space isomorphisms. By definition f_α is also a vector space isomorphism, because $\sigma_\alpha \in \mathrm{GL}(k, \mathbb{R})$. Finallly, we claim that F is a bijection. Let $\widetilde{E}_q$ be a fiber over q. Then E_q is a fiber over q, so there exists $q, v \in E$. Then

$$\widetilde{\pi} \circ F(q, v) = \widetilde{\pi}(q, Av) = q$$

for some matrix A. Thus the image of F includes each fiber $\widetilde{E}_q$. Since F is a linear isomorphism on each fiber, this shows that F is a bijection. Then by Proposition 10.26, this makes F a bundle isomorphism. □

Lemma 0.3 (for Exercise 11-6)**.** *Let M be a smooth manifold and $p \in M$ and $\lambda \in T_p^*M$. Then there exists a neighborhood U of p and a smooth function $y : U \to \mathbb{R}$ such that $dy|_p = \lambda$.*

Proof. Let $(U, (x^i))$ be a smooth chart with $p \in U$. Then let $(\frac{d}{dx^i}|_p)$ be the usual basis for T_pM and $dx^i|_p$ be the dual basis for T_p^*M. Then we can write λ as

$$\lambda = \sum_i \lambda_i dx^i|_p$$

for scalars $\lambda_i \in \mathbb{R}$. Define $y = \sum_i \lambda_i x^i$. Then

$$dy|_p = d\left(\sum_i \lambda_i x^i\right) = \sum_i \lambda_i dx^i|_p = \lambda$$

□

Proposition 0.4 (Exercise 11-6)**.** *Let M be a smooth n-manifold, $p \in M$, and $y^1, \dots y^k$ smooth real-valued funcions on a neighborhood of p. Then*

1. *If $k = n$ and $(dy^1|_p, \dots, dy^n|_p)$ is a basis for T_p^*M then $(y^1, \dots, y^n)$ are smooth coordinates for M in a neighborhood of p.*
2. *If $(dy^1|_p, \dots, dy^k|_p)$ is a linearly independent k-tuple of covectors and $k < n$, then there are smooth functions $y^{k+1}, \dots y^n$ such that $(y^1, \dots y^n)$ are smooth coordinates for M in a neighborhood of p.*
3. *If $(dy^1|_p, \dots, dy^k|_p)$ span T_p^*M (this implies $k > n$), there are indices $i_1, \dots i_n$ such that $(y^{i_1}, \dots, y^{i_n})$ are smooth coordinates for M in a neighborhood of p.*

Proof. First we show (1). Let $(y^1, \dots, y^n)$ be smooth real-valued functions on a neighborhood U of p, such that $(dy^1|_p, \dots, dy^n|_p)$ is a basis for T_p^*M. Define $F : U \to \mathbb{R}^n$ by $F(p) = (y^1(p), \dots, y^n(p))$. We claim that dF_p is invertible. Let $(v_1, \dots, v_n)$ be the dual basis to $(dy^1|_p, \dots, dy^n|_p)$, so

$$v_i(dy^j|_p) = \delta_{ij}$$

We canonically identify $(T_p^*M)^*$ with T_pM, so we can also think of v_i as a vector in T_pM. If $(x^1, \dots, x^n)$ are the standard coordinate functions on $\mathbb{R}^n$, then we have

$$dF_p(v_i)(x^j) = v_i(x^j \circ F) = v_i y^j = \delta_{ij}$$

We claim that the kernel of dF_p is trivial. If $a^i v_i \in \ker dF_p$, then

$$0 = dF_p(a^i v_i)(x^j) = \sum_i a^i \delta_{ij} = a^j$$

for all j, so $a^i v_i = 0$. Thus dF_p has trivial kernel, so it is injective. Since it is a linear map between T_pM and $T_p\mathbb{R}^n$, vector spaces of the same dimension, this implies that it is bijective (and hence invertible). Now by Theorem 4.5 (Inverse Function Theorem for Manifolds), there exists a neighborhood V of p such that $F|_V : V \to F(V) \subset \mathbb{R}^n$ is a diffeomorphism. Thus $(y^1, \dots y^n)$ are smooth local coordinates on V.

Now we show (2). We have a linearly independent k-tuple of covectors $(dy_p^1, \dots, dy^k|_p)$ in T_p^*M, so we can extend it to a basis $(dy^1|_p, \dots dy^k|_p, \omega^{k+1}, \dots \omega^n)$. By the above lemma, there exist smooth functions $y^{k+1}, \dots, y^n$ defined on a neighborhood of p such that $dy^{k+1}|_p = \omega^{k+1}, \dots, dy^n|_p = \omega^n$. Then by part (1), $(y^1, \dots, y^n)$ are smooth coordinates for M in a neighborhood of p.

Now we show (3). We have $(dy^1|_p, \dots, dy^k|_p)$ spanning T_p^*M. As every spanning set of a vector space contains a basis, there exist indices $i_1, \dots i_n$ such that $(dy^{i_1}|_p, \dots, dy^{i_n}|_p)$ is a basis for T_p^*M. The by part (1), $(y^{i_1}, \dots, y^{i_n})$ are smooth coordinates for M in a neighborhood of p. □

Proposition 0.5 (Exercise 11-7a)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $F(x, y) = (xy, e^y) = (u, v)$ and let $\omega_{(x,y)} = x\ dy - y\ dx$. Then*

$$F^*\omega = (x - 1)ye^x\ dx - xe^x\ dy$$

Proof.

$$\begin{aligned} F^*\omega &= (u\circ F)d(v\circ F) + (-v\circ F)d(u\circ F) = xyd(e^x) - e^x d(xy) \\ &= xye^x\, dx - e^x(y\, dx + x\, dy) = xye^x\, dx - ye^x\, dx - xe^x\, dy \\ &= (x-1)ye^x\, dx - xe^x\, dy \end{aligned}$$

□

Proposition 0.6 (Exercise 11-10c). *Let $f : S^2 \to \mathbb{R}$ be the map $f(p) = z(p)$, the z-coodinate for p as a point in $\mathbb{R}^3$. Let (u, v) be the stereographic coordinates on $S^2 \setminus \{N\}$. Then*

$$df = \frac{4u}{(1+u^2+v^2)^2}du + \frac{4v}{(1+u^2+v^2)^2}dv$$

on $S^2 \setminus \{N\}$, and $df_p = 0$ at only the north and south poles.

Proof. We can write f as

$$f(u, v) = \frac{u^2+v^2-1}{u^2+v^2+1}$$

Then

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \frac{4u}{(1+u^2+v^2)^2}du + \frac{4v}{(1+u^2+v^2)^2}dv$$

This is zero precisely when $(u, v) = (0, 0)$ which is at the south pole. Let (s, t) be the stereographic coordinates on $S^2 \setminus \{S\}$, then we have

$$f(s, t) = \frac{-s^2-t^2+1}{s^2+t^2+1}$$

then

$$df = \frac{-4s}{(1+s^2+t^2)^2}ds + \frac{-4t}{(1+s^2+t^2)^2}dt$$

This agrees with our other computation of df on $S^2 \setminus \{N, S\}$ and allows us to compute $df_N = df_{(s,t)=(0,0)} = 0$. □

Proposition 0.7 (Exercise 11-11). *Let M be a smooth n-manifold and $C \subset M$ an embedded k-dimensional submanifold. Let $f \in C^\infty(M)$ and suppose that $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Let $\phi : U \to \mathbb{R}^k$ be a smooth local defining function for C on a neighborhood U of p in M. Then there are real numbers $\lambda_1, \ldots \lambda_k$ such that*

$$df_p = \lambda_1 d\phi^1|_p + \ldots + \lambda_k d\phi^k|_p$$

Proof. By Theorem 5.8, there is a smooth chart (V, ψ) for M with $p \in V$ such that $V \cap C$ is a single k-slice in U. We may choose $V \subset U$ by taking the portion of V contained in U if necessary. Let $\widehat{f} = f \circ \psi^{-1}$ be the coordinate representation of f, and define $\widehat{V} = \psi(V)$ and $\widehat{\phi} = \phi \circ \psi^{-1}$ and $\widehat{p} = \psi(p)$. Note that then $\widehat{p}$ is a local extrema for $\widehat{f}$.

Because ϕ is a defining function for C, it is constant on C, so $\widehat{\phi}$ is constant on $\widehat{V}$. If $\phi^1, \ldots, \phi^k$ are the component functions of ϕ, then we have functions $\widehat{\phi^i} : \widehat{V} \to \mathbb{R}$, all of which are constant functions as $\widehat{\phi}$ is constant on $\widehat{V}$. Let $c_i = \widehat{\phi}^i(x)$. So $\widehat{p}$ is a local extrema of $\widehat{f}$ subject to the constraints

$$\widehat{\phi}^i(x) - c_i = 0$$

Because $\widehat{\phi}$ is a submersion, the Jacobian is invertible. Thus by the method of Lagrange multipliers on $\mathbb{R}^n$, we know that there are real constants $\lambda_1, \ldots, \lambda_k$ so that

$$d\widehat{f}_{\widehat{p}} = \lambda_j d\widehat{\phi}^j|_{\widehat{p}}$$

By the chain rule, we have

$$\begin{aligned} d\widehat{f}_{\widehat{p}} &= d(f \circ \psi^{-1})_{\widehat{p}} = df_p \circ d\psi^{-1}_{\widehat{p}} \\ d\widehat{\phi}^j &= d(\phi^j \circ \psi^{-1})_{\widehat{p}} = d\phi^j_p \circ d\psi^{-1}_{\widehat{p}} \end{aligned}$$

so then

$$df_p \circ d\psi^{-1}_{\widehat{p}} = \lambda_j (d\phi^j_p \circ d\psi^{-1}_{\widehat{p}}) \implies df_p = \lambda_j d\phi^j_p \circ d\psi^{-1}_{\widehat{p}} \circ d\psi_{\widehat{p}} = \lambda_j d\phi^j_p$$

which was what we set out to show. □